Isometric immersions of homogeneous spaces (Lie Groups, Geometric Structures and Differential Equations : One Hundred Years after Sophus Lie)

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Isometric immersions of homogeneous spaces

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Abstract

In the theory of isometric immersions of submanifolds there are fundamental theorems of John Nash for the C^r-case and Burstin-Cartan-Janet-Schläfly for the analytic case (also see Robert Greene (2) for the case of local isometric immersions). These theorems require, however, a large codimension and are of practically no help in considering concrete questions in low codimensions. An obvious way of producing large varieties of isometrically immersed homogeneous submanifolds is to take the orbits of Lie group actions. In low codimensions the following theorem should often be true: Let a compact, connected Lie group G act on the connected manifold N with principal orbit type $M = G/H$. Then, among all the G-homogeneous metrics on G/H the only ones which allow an isometric immersion into N are those which are already realized as the orbit metrics of this action. Obviously this is true for the spheres $S^{n-1}(r)$ of $\mathbb{R}^n$ under the standard $SO(n)$-action. With a little work it is also easy to prove for the larger classes of metrics invariant under the unitary or symplectic groups. A slightly more challenging example is to prove this

result for the second Stiefel manifold of 2-frames: $SO(n)/SO(n-2)$ of $\mathbb{R}^{2n}$ under the diagonal embedding $SO(n) \to SO(n) \times SO(n)$ acting on $\mathbb{R}^{2n}$.

Such theorems follow from a careful study of the Gauss equation: $\langle \hat{R}(X \wedge Y)Z, W\rangle = B(X,W)B(Y,Z) - B(X,Z)B(Y,W)$, where $\hat{R}$ is the curvature operator of the submanifold M and B is the second fundamental form. In the case of a non-Euclidean surrounding space N, however, the Gauss equation reads: $\langle R^t(X \wedge Y)Z, W\rangle = B(X,Z)B(Y,W) - B(X,W)B(Y,Z)$, where $R^t = \bar{R}^t - \hat{R}$ and $\bar{R}^t$ is the part of the curvature operator $\bar{R}$ of N tangential to M. This varies with M's position in N, so the left hand side is also not given. There has not been much study of this, but we will report on the solution of the probably most basic question in isometric immersions into non-classical geometries.

Let $N = \mathbb{C}P(n)$ with metric normalized such that sectional curvatures are in $[1,4]$. Each geodesic sphere $S^{2n-1}(r) = U(n)/U(n-1)$ determines a different homothety class, $\alpha^2\gamma_r$ of metrics $(r \in \left(0, \frac{\pi}{2}\right), \alpha^2 \in (0,\infty))$.

Theorem: *These Berger metrics $\gamma_r(r \in (0, \frac{\pi}{2}))$ are the only $U(n)$-invariant metrics on S^{2n-1} which allow an isometric immersion into $\mathbb{C}P(n)$.*

Remark: This is a 1-dimensional set in the 2-dimensional variety of all $U(n)$-imvariant metrics. Thus no non-trivial homothetic image $\alpha^2\gamma_r$ of γ_r $(\alpha^2 \neq 1)$ or no Berger metric from a geodesic sphere in complex hyperbolic space allows such an isometric immersion.

We give the flavor of the argument for one of the easier cases:

Let $\{e_i, Je_i; i = 0, \dots, n-1\}$ be a adapted orthonormal basis for $T_p\mathbb{C}P(n) \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$. Then

$$\bar{R}(e_i \wedge e_j) = -e_i \wedge e_j - Je_i \wedge Je_j$$

$$\bar{R}(e_i \wedge Je_i) = -2e_0 \wedge Je_0 - \cdots - 4e_i \wedge Je_i - \cdots - 2e_{n-1} \wedge Je_{n-1}$$

For $S^{2n-1} = U(n)/U(n-1)$ we choose an adapted orthonormal basis $Y_0, Y_1, J'Y_1, \ldots, Y_{n-1}$, $J'Y_{n-1}$. $T_pJ'Y_{n-1}T_p, S^{2n-1} \cong \mathbb{R}^{2n-1} = \mathbb{R}Y_0 \oplus \mathbb{R}^{2n-2}$ where the isotropy action of $U(n-1)$ on $\mathbb{R}^{2n-2}$ is by the standard representation and J' is the almost complex structure defined by this action on $\mathbb{R}^{2n-2}$. Now, let $\alpha^2\gamma_r$ be the metric of $S^{2n-1}(r)$ multiplied by α^2. Then

$$\begin{aligned}
\hat{R}(Y_0 \wedge Y_i) &= -\frac{\cot^2 r}{\alpha^4} Y_0 \wedge Y_i \\
\hat{R}(Y_i \wedge Y_j) &= -\frac{1+\cot^2 r}{\alpha^4} Y_i \wedge Y_j - \frac{1}{\alpha^4} J'Y_i \wedge J'Y_j \\
\hat{R}(Y_i \wedge J'Y_i) &= -\frac{2}{\alpha^4} Y_1 \wedge J'Y_1 \\
&\quad - \cdots - \frac{4+\cot^2 r}{\alpha^4} Y_i \wedge J'Y_i - \cdots - \frac{2}{\alpha^4} Y_{n-1} \wedge J'Y_{n-1} \,.
\end{aligned}$$

Now, choose the basis $\{e_i, Je_i\}$ such that Je_0 is normal to M at p. Then $Y_0 = e_0 \cos\varphi + Je_1 \sin\varphi$, and let $Y_1 = e_1$.

Lemma 1 $R^t(X \wedge Y) \wedge R^t(X \wedge Z) = 0$.

Proof. The Gauss equation may be written: $\langle R^t(X \wedge Y)W_1, W_2\rangle = -(B \wedge B)(X, Y, W_1, W_2)$ $= -\widetilde{B \wedge B}(X \wedge Y)(W_1, W_2)$ where $B \wedge B$ is defined by: $(B \wedge B)(X, Y, Z, W) = B(X, Z)$ $B(Y, W) - B(X, W)B(Y, Z)$. Let $\hat{B}$ be the shape operator, i.e. $\langle \hat{B}(X), Y\rangle = B(X, Y)$. Then: $R^t(X \wedge Y) = \hat{B}(X) \wedge \hat{B}(Y)$. Hence $R^t(X \wedge Y) \wedge R^t(X \wedge Z) = \hat{B}(X) \wedge \hat{B}(Y) \wedge \hat{B}(X) \wedge \hat{B}(Z) = 0$. q.e.d.

Proposition 2 *We have* $\sin\varphi = 0$, *hence we may assume* $Y_0 = e_0$.

Proof. $R^t(Y_0 \wedge Y_1) = \cos\varphi(-1 + \frac{\cot^2 r}{\alpha^4})e_0 \wedge e_1 + \sin\varphi[(4 - \frac{\cot^2 r}{\alpha^4})e_1 \wedge Je_1 + 2e_2 \wedge Je_2 + \cdots + 2e_{n-1} \wedge Je_{n-1}]$. For $n \geq 4$ it is easy: $R^t(Y_0 \wedge Y_1) \wedge R^t(Y_0 \wedge Y_1) = \cdots 8\sin^2\varphi e_2 \wedge Je_2 \wedge e_3 \wedge Je_3 + \cdots = 0$, i.e. $\sin^2\varphi = 0$. For $n = 3$ it follows by more delicate choices. q.e.d.

Proposition 3 *We have* $\alpha^4 = 1$, *i.e.* $S^{2n-1} = S^{2n-1}(r)$.

Proof. We have $Y_0 = e_0$, $Y_1 = e_1$, $J'Y_1 = \cos\varphi J e_1 + \sin\varphi e_2$. Now choose $Y_3 = e_3$ perpendicular to $e_0, e_1, Je_1, e_2, Je_2$ (and hence to $Y_0, Y_1, J'Y_1$). Then $R^t(Y_1\wedge Y_3) = \bar{R}^t(e_1\wedge e_3) - \hat{R}(Y_1\wedge Y_3) = (-1+\frac{1+\cot^2 r}{\alpha^4})e_1\wedge e_3 - Je_1\wedge Je_3 + \frac{1}{\alpha^4}J'e_1\wedge J'e_3$. $R^t(Y_1\wedge Y_3)\wedge R^t(Y_1\wedge Y_3) = 2(-1+\frac{1+\cot^2 r}{\alpha^4})e_1\wedge e_3\wedge(-Je_1\wedge Je_3 + \frac{1}{\alpha^4}J'e_1\wedge J'e_3) = 0$. $R^t(Y_0\wedge Y_1)\wedge R^t(Y_1\wedge Y_3) = 2(-1+\frac{\cot^2 r}{\alpha^4})e_0\wedge e_1\wedge(-Je_1\wedge e_3 + \frac{1}{\alpha^4}J'e_1\wedge J'e_3)$. Hence $Je_1\wedge Je_3 = \frac{1}{\alpha^4}J'e_1\wedge J'e_3$, and $\alpha^4 = 1$. q.e.d.

We now deal with the more complicated cases $n = 3$ and (especially) $n = 2$, and give a couple of typical arguments. Assume first $n = 3$. Then $R^t(Y_0\wedge Y_1)\wedge R^t(Y_0\wedge Y_1) = 2\sin\varphi\cos\varphi\big(\frac{\cot^2 r}{\alpha^4} - 1\big)e_0\wedge e_1\wedge e_2\wedge Je_2 + 2\sin^2\varphi\big(4 - \frac{\cot^2 r}{\alpha^4}\big)e_1\wedge Je_1\wedge e_2\wedge Je_2 +$ terms of other type $= 0$. Hence, either $\sin\varphi = 0$ or $\frac{\cot^2 r}{\alpha^4} = 4$. In the latter case we have the term $6\sin\varphi\cos\varphi e_0\wedge e_1\wedge e_2\wedge Je_2$, hence $\cos\varphi = 0$; i.e. we let $Y_0 = Je_1$, $Y_1 = e_1$. Choose $Y_2 = e_2$ orthonormal to $e_0, e_1, Je_1, J'Y_1$ (and hence to Y_1, Y_0). Then $R^t(Y_0\wedge Y_2) = \bar{R}^t(Je_1\wedge e_2) - \hat{R}^t(Y_0\wedge Y_2) = -Je_1\wedge e_2 + e_1\wedge Je_2 + \frac{\cot^2 r}{\alpha^4}Y_0\wedge Y_2 = \big(\frac{\cot^2 r}{\alpha^4} - 1\big)Je_1\wedge e_2 + e_1\wedge Je_2$. Hence $R^t(Y_0\wedge Y_2)\wedge R^t(Y_0\wedge Y_2) = 2\big(\frac{\cot^2 r}{\alpha^4} - 1\big)Je_1\wedge e_2\wedge e_1\wedge Je_2 = 6Je_1\wedge e_2\wedge e_1\wedge Je_2 \neq 0$, which is a contradition. Hence $\sin\varphi = 0$.

This proves the following:

Proposition 4 *For $n = 3$ we have $\sin\varphi = 0$, i.e. $Y_0 = e_0$.*

We also need:

Proposition 5 *For $n = 3$ we have $\alpha^2 = 1$.*

Proof. $Y_0 = e_0$, $Y_1 = e_1$, $J'Y_1 = \cos\psi Je_1 + \sin\psi e_2$. We define: $Y_2 = -\sin\psi Je_1 + \cos\psi e_2$, then Y_2 is orthonormal to $Y_0, Y_1, J'Y_1$. By dimension $J'Y_2 = \pm Je_2$. $Y_2\wedge J'Y_1 = -Je_1\wedge e_2$, hence: $R^t(Y_1\wedge J'Y_2) = \bar{R}^t(e_1\wedge(\pm Je_2)) - \hat{R}(Y_1\wedge J'Y_2) = \mp e_1\wedge Je_2 \pm Je_1\wedge e_2 + \frac{1+\cot^2 r}{\alpha^4}Y_1\wedge J'Y_1 - \frac{1}{\alpha^4}J'Y_1\wedge Y_2 = \big(\frac{1+\cot^2 r}{\alpha^4} - 1\big)Y_1\wedge J'Y_2 + \big(\pm 1 - \frac{1}{\alpha^4}\big)J'Y_1\wedge Y_2$. By setting

$R^t(Y_1 \wedge J'Y_2) \wedge R^t(Y_1 \wedge J'Y_2) = 0$ we get: a) $1 + \cot^2 r = \alpha^4$ or b) $\alpha^4 = 1$. We also have $R^t(Y_0 \wedge Y_1) = \left(\frac{\cot^2 r}{\alpha^4} - 1\right) Y_0 \wedge Y_1$. In case a) we have $R^t(Y_0 \wedge Y_1) \wedge R^t(Y_1 \wedge J'Y_2) = \left(-\frac{1}{\alpha^4}\right)\left(\pm 1 - \frac{1}{\alpha^4}\right) Y_0 \wedge Y_1 \wedge J'Y_1 \wedge Y_2 \neq 0$, which is a contradition. Hence $\alpha^2 = 1$.

q.e.d.

For $n = 2$ this argument breaks down, since $\dim M_p = 3$ and any wedge product of 4 vectors is zero. Indeed, the Gauss equations do have other solutions. Most of those are eliminated by the Codazzi equations: $\langle R(X \wedge Y)Z, N\rangle = YB(X,Z) - B(X, \nabla_Y Z) - XB(Y,Z) + B(Y, \nabla_X Z) + B([X,Y], Z)$. We note that this is a considerably more complicated case and only outline a few highlights of the constructions.

Let $n = 2$ and let $SU(2) \cong S^3(r) \subset \mathbb{C}P(2)$ be the geodesic sphere of radius r, $r \in \left(0, \frac{\pi}{2}\right)$. An orthogonal basis for $SU(2)$ at a point p is given by: $E_0 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $E_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $E_2 = J'E_1 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. An orthonormal basis is given by $\frac{1}{\sin r \cos r} E_0$, $\frac{1}{\sin r} E_1$, $\frac{1}{\sin r} E_2$, and a homothetic image of this by: $Y_0 = \frac{1}{\alpha^2 \sin r \cos r} E_0$, $Y_i = \frac{1}{\alpha^2 \sin r} E_i$, $i = 1, 2$. We again need to prove that if S^3, with Y_i as an orthonormal basis, admits an isometric immersion into $\mathbb{C}P(2)$, it follows that $\alpha^2 = 1$.

Extend Y_i to left-invariant vector fields on S^3. Then, in the Koszul formula: $\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle$ the three first terms of the right hand side vanish.

We compute:

$$(1) \qquad [Y_0, Y_1] = \frac{2}{\alpha^2 \sin r \cos r} Y_2, \quad [Y_0, Y_2] = -\frac{2}{\alpha^2 \sin r \cos r} Y_1, \quad [Y_1, Y_2] = \frac{2 \cot r}{\alpha^2} Y_0$$

and by repeated use of the Koszul formula we find:

$$\nabla_{Y_0} Y_1 = \frac{1+\sin^2 r}{\alpha^2 \sin r \cos r} Y_2, \qquad \nabla_{Y_1} Y_0 = -\frac{\cot r}{\alpha^2} Y_2,$$
$$\nabla_{Y_0} Y_2 = -\frac{1+\sin^2 r}{\alpha^2 \sin r \cos r} Y_1, \qquad \nabla_{Y_2} Y_0 = \frac{\cot r}{\alpha^2} Y_1,$$
$$\nabla_{Y_1} Y_2 = \frac{\cot r}{\alpha^2} Y_0, \qquad \nabla_{Y_2} Y_1 = -\frac{\cot r}{\alpha^2} Y_0$$
$$\nabla_{Y_0} Y_0 = \nabla_{Y_1} Y_1 = \nabla_{Y_2} Y_2 = 0\,. \tag{2}$$

$\hat{R}(Y_0 \wedge Y_1)Y_0 = \nabla_{Y_0}\nabla_{Y_1}Y_0 - \nabla_{Y_1}\nabla_{Y_0}Y_0 - \nabla_{[Y_0,Y_1]}Y_0 = -\nabla_{Y_0}\left(\frac{\cot r}{\alpha^2}Y_2\right) - 0 - \frac{2}{\alpha^2 \sin r \cos r}\frac{\cot r}{\alpha^2}\nabla_{Y_2}Y_0 = \frac{\cot r}{\alpha^2}\frac{1+\sin^2 r}{\alpha^2 \sin r \cos r}Y_1 - \frac{\cot r}{\alpha^2}\frac{2}{\alpha^2 \sin r \cos r}Y_1 = -\frac{\cot^2 r}{\alpha^4}Y_1.$

Similarly: $\hat{R}(Y_0 \wedge Y_1)Y_1 = \frac{\cot^2 r}{\alpha^4}Y_0$ and $\hat{R}(Y_0 \wedge Y_1)Y_2 = 0$. Hence $\hat{R}(Y_0 \wedge Y_1) = -\frac{\cot^2 r}{\alpha^4}Y_0 \wedge Y_1$. By similar computations $\hat{R}(Y_0 \wedge Y_2) = -\frac{\cot^2 r}{\alpha^4}Y_0 \wedge Y_2$ and $\hat{R}(Y_1 \wedge Y_2) = -\frac{4+\cot^2 r}{\alpha^4}Y_1 \wedge Y_2$. Now, let $-Je_0$ be normal to S^3 at p, then $J(-Je_0) = e_0$ is in T_pS^3. $Y_0 = e_0 \cos\varphi + Je_1 \sin\varphi$. Choose $Y_1 = e_1$, then Y_1 is normal to e_0 and Y_0. $Y_2 = \mp(\sin\varphi e_0 - \cos\varphi Je_1)$, say $Y_2 = -\sin\varphi e_0 + \cos\varphi Je_1$. Now $\bar{R}(e_0 \wedge e_i) = -e_0 \wedge e_i - Je_0 \wedge Je_i$, $i = 1, 2$.

$\bar{R}(e_1 \wedge e_2) = -4e_1 \wedge e_2 - 2e_0 \wedge Je_0$.

$R(Y_0 \wedge Y_1) = \bar{R}(Y_0 \wedge Y_1) - \hat{R}(Y_0 \wedge Y_1) = \bar{R}((e_0 \cos\varphi + Je_1 \sin\varphi) \wedge e_1) + \frac{\cot^2 r}{\alpha^4}Y_0 \wedge Y_1$
$= \left(\frac{\cot^2 r}{\alpha^4} - 1 - 3\sin^2\varphi\right)Y_0 \wedge Y_1 + 3\sin\varphi\cos\varphi Y_1 \wedge Y_2 + [3\sin\varphi\cos\varphi Y_0 - (3\sin^2\varphi - 1)Y_2] \wedge Je_0$.

By similar computations $R(Y_0 \wedge Y_2) = \left(\frac{\cot^2 r}{\alpha^4} - 1\right)Y_0 \wedge Y_2 - Y_1 \wedge Je_0$, and $R(Y_1 \wedge Y_2) = 3\sin\varphi\cos\varphi Y_0 \wedge Y_1 + \left(\frac{\cot^2 r + 4}{\alpha^4} - 1 - 3\cos^2\varphi\right)Y_1 \wedge Y_2 + [3\sin\varphi\cos\varphi Y_2 + (1 - 3\cos^2\varphi)Y_0] \wedge Je_0$.

Now, consider Gauss' equations:

G1. $\langle R(Y_0 \wedge Y_1)Y_0, Y_1\rangle = \frac{\cot^2 r}{\alpha^4} - 1 - 3\sin^2\varphi = b_{00}b_{11} - b_{01}^2$

G2. $\langle R(Y_0 \wedge Y_1)Y_0, Y_2\rangle = 0 = b_{00}b_{12} - b_{01}b_{02}$

G3. $\langle R(Y_0 \wedge Y_1)Y_1, Y_2\rangle = 3\sin\varphi\cos\varphi = b_{01}b_{12} - b_{02}b_{11}$

G4. $\langle R(Y_0 \wedge Y_2)Y_0, Y_2\rangle = \frac{\cot^2 r}{\alpha^4} - 1 = b_{00}b_{22} - b_{02}^2$

G5. $\langle R(Y_1 \wedge Y_2)Y_0, Y_2\rangle = 0 = b_{01}b_{22} - b_{02}b_{12}$

G6. $\langle R(Y_1 \wedge Y_2)Y_1, Y_2\rangle = \frac{4+\cot^2 r}{\alpha^4} - 1 - 3\cos^2\varphi = b_{11}b_{22} - b_{12}^2$ where $b_{ij} = B(Y_i, Y_j)$

For example, by expanding the determinant $\begin{vmatrix} b_{00} b_{01} b_{02} \\ b_{00} b_{01} b_{02} \\ b_{02} b_{12} b_{22} \end{vmatrix} = 0$ after the first line, I obtain:

$$0 \cdot b_{00} - \Big(\frac{\cot^2 r}{\alpha^4} - 1\Big) b_{01} + 0 \cdot b_{02} = 0 \, .$$

Assume $\alpha^4 \neq \cot^2 r$, then $b_{01} = 0$. Similarly, under this assumption, we get:

$$\begin{aligned} &b_{12} = 0 \, . \\ &-3 \sin\varphi \cos\varphi b_{00} + \Big(1 + 3 \sin\varphi - \frac{\cot^2 r}{\alpha^4}\Big) b_{02} = 0 \\ &\Big(\frac{4 + \cot^2 r}{\alpha^4} - 1 - 3\cos^2\varphi\Big) b_{02} + 3 \sin\varphi \cos\varphi b_{22} = 0 \end{aligned} \tag{3}$$

Furthermore: $\det B = \begin{vmatrix} b_{00} b_{01} b_{02} \\ b_{01} b_{11} b_{12} \\ b_{02} b_{12} b_{22} \end{vmatrix}$

$$\begin{aligned} &\Big(\frac{4 + \cot^2 r}{\alpha^4} - 1 - 3\cos^2\varphi\Big) b_{00} + 3 \sin\varphi \cos\varphi b_{02} \\ &\quad = \Big(\frac{\cot^2 r}{\alpha^4} - 1\Big) b_{11} = 3 \sin\varphi \cos\varphi b_{02} + \Big(\frac{\cot^2 r}{\alpha^4} - 1 - 3\sin^2\varphi\Big) b_{22} \, . \end{aligned} \tag{4}$$

(3) and (4) give 4 equations for the 4 unknowns $b_{00}, b_{11}, b_{22}, b_{02}$ (in terms of φ). But we also have the Codazzi equaions: $\langle R(X \wedge Y)Z, N\rangle = YB(X,Z) - B(X, \nabla_Y Z) - XB(Y,Z) + B(Y, \nabla_X Z) + B([X,Y],Z)\rangle$, where N is a normal vector of $T_p S^3$.

C1. $\langle R(Y_0 \wedge Y_1)Y_0, -Je_0\rangle = -3 \sin\varphi \cos\varphi$

$$= Y_1(b_{00}) + \tfrac{\cot r}{\alpha^2} b_{02} + \tfrac{2}{\alpha^2 \sin r \cos r} b_{02} = Y_1(b_{00}) + \tfrac{2 + \cos^2 r}{\alpha^2 \sin r \cos r} b_{02}$$

Similarly:

C5. $\langle R(Y_0 \wedge Y_2)Y_1, -Je_0\rangle = 1 = \frac{\cot r}{\alpha^2} b_{00} - \frac{2}{\alpha^2 \sin r \cos r} b_{11} + \frac{1 + \sin^2 r}{\alpha^2 \sin r \cos r} b_{22}$

C9. $\langle R(Y_1 \wedge Y_2)Y_2, -Je_0\rangle = -3 \sin\varphi \cos\varphi = -Y_1(b_{22}) + 3\frac{\cot r}{\alpha^2} b_{02}$

(There are 9 such equations altogether.)

Now, from the 4 equations from (3) and (4) we may solve b_{00}, b_{11}, b_{22} (in terms of φ) and substitute into C5. Although this is quite laborious, it works, and we get a non-trivial expression in $\sin\varphi$ equal to 1. Hence $\sin\varphi$ must be constant, and the terms $Y_1(b_{00}), Y_1(b_{22})$ must vanish. C1 and C9 gives $\frac{2 + \cos^2 r}{\alpha^2 \sin r \cos r} b_{02} = \frac{3 \cos^2 r}{\alpha^2 \sin r \cos r} b_{02}$, since $r \in \big(0, \frac{\pi}{2}\big)$ this implies $b_{02} = 0$. But then, by C1 again, $\sin\varphi \cos\varphi = 0$. One can show that $\cos\varphi = 0$ leads to a contradiction. Consider $\sin\varphi = 0$. Then $b_{00}b_{11} = \frac{\cot^2 r}{\alpha^4} - 1 = b_{00}b_{22}$. Since $\alpha^4 \neq \cot^2 r$, $b_{11} = b_{22} \neq 0$. $b_{11}b_{22} = b_{11}^2 = \frac{4+\cot^2 r}{\alpha^4} - 4$. C5 now says: $\frac{\cot r}{\alpha^2}(b_{00} - b_{11}) = 1$. By solving $b_{11} = \pm\sqrt{\frac{4+\cot^2 r}{\alpha^4} - 4}$ and $b_{00} = \pm\left(\frac{\cot^2 r}{\alpha^4} - 1\right)\left(\frac{4+\cot^2 r}{\alpha^4} - 4\right)^{-1/2}$ and substituting this into C5 we now get an equation for α:

$$\alpha^{12} + (2\cot^2 r - 1)\alpha^8 - 6\cot^2 r\alpha^4 + 4\cot^2 r = 0\,.$$

Obviously $\alpha^4 = 1$ is a solution, and by dividing by $\alpha^4 - 1$ we get: $\alpha^8 + 2\cot^2 r\alpha^4 - 4\cot^2 r = 0$. $\alpha^4 = \sqrt{\cot^4 r + 4\cot^2 r} - \cot^2 r$ as a possible second solution. To eliminate this we consider Maeda's condition for the principal curvatures of a hypersurface. b_{00} is the principal curvature along the principal direction, b_{11} along the direction e_1. Then $b_{22} = \frac{b_{00}b_{11}+2}{2b_{11}-b_{00}} = b_{11}$, i.e. $b_{00}b_{11}+2 = 2b_{11}^2 - b_{00}b_{11}$. $2b_{11}^2 - 2b_{00}b_{11} = \frac{8+2\cot^2 r}{\alpha^4} - 8 - 2\frac{\cot^2 r}{\alpha^4} + 2 = 2$ $\frac{8}{\alpha^4} = 8$, i.e. $\alpha^4 = 1$. q.e.d.

Remark. Alternatively, we could require that the curvature tensor of the total curvature constructed from the metric of S^3 and B be parallell.

Remark. This is of course only a small part of the work. One needs to check $\cos\varphi = 0$, $\alpha^4 = \cot^2 r$, and then one must check that none of the $U(n)$-invariant metrics of the spheres in complex hyperbolic or Euclidean space admit an isometric immersion into $\mathbb{C}P(n)$. All these results are true for local isometric immersions, obviously. Corresponding results also hold for complex hyperbolic space and quaternionic projective and hyperbolic spaces, and will be published shortly.

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